Middle East Technical University

Department of Mechanical Engineering

ME 413 Introduction to Finite Element Analysis

Chapter 2

Introduction to FEM

These notes are prepared by

Dr. Cüneyt Sert

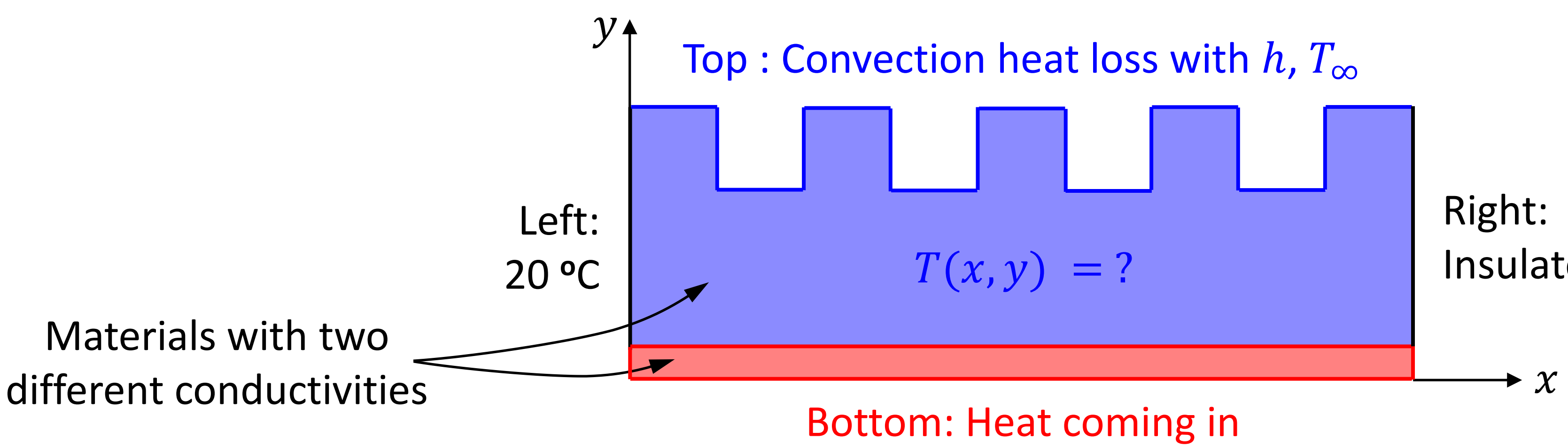
http://www.me.metu.edu.tr/people/cuneyt

csert@metu.edu.tr

Disadvantages of Ritz and MWR

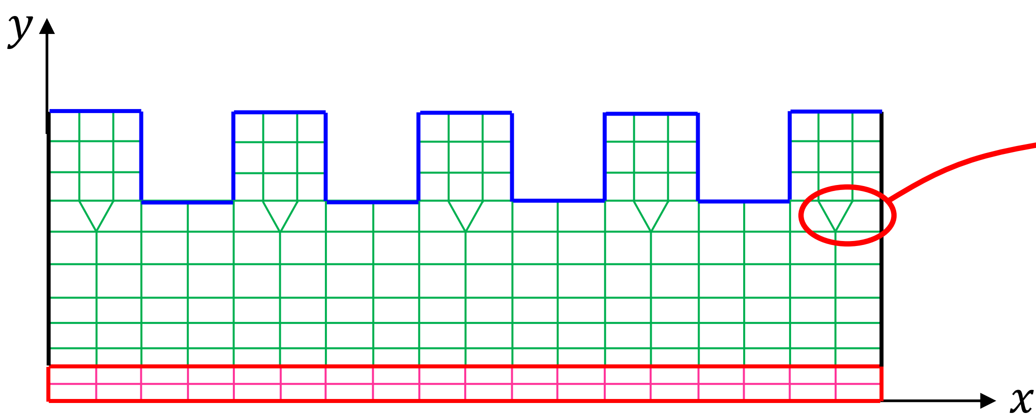
- They provide global solutions, i.e. a single approximate solution is valid over the whole problem domain.
  - Difficult to capture complicated 2D and 3D solutions on complex domains.
  - Not suitable to solve problems with multiple materials.

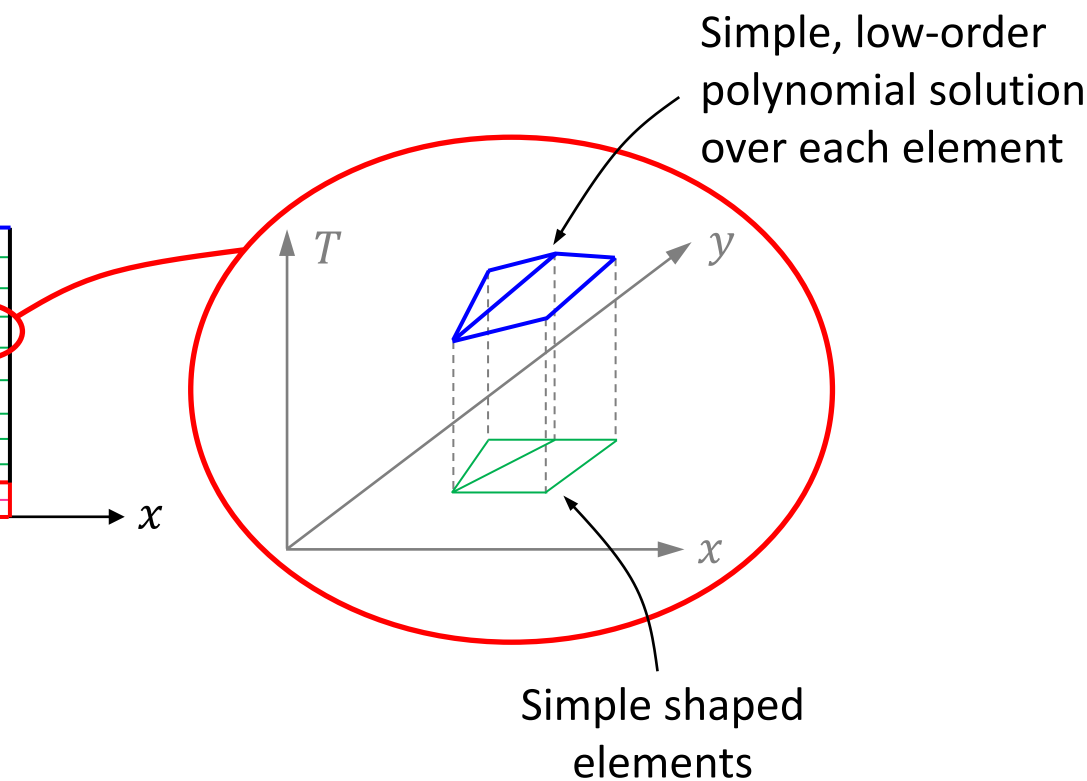
- Approximation function selection is
  - problem (DE, BC, domain size) dependent. Difficult to automate.
  - practically impossible for complex 2D and 3D geometries. See the problem below.
  - NOT unique.



FEM vs. Ritz

- Finite Element Method (FEM)
  - does NOT seek a global solution
  - divides the problem domain into elements of simple shapes
  - works with simple polynomial type approximate solutions over each element





- FEM is similar to Ritz method
  - uses weak form
  - weight function selection is the same

Our First FE Solution

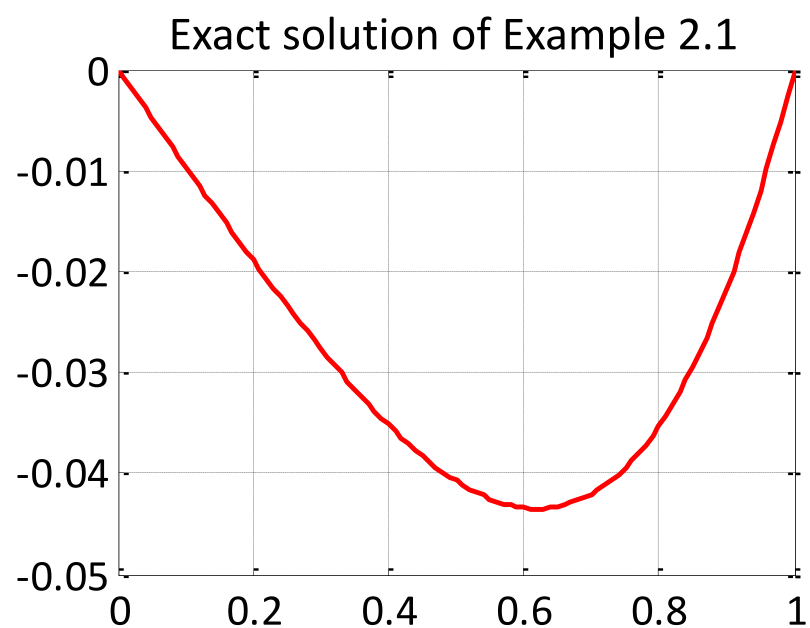
e.g. **Example 2.1** Solve the following problem using FEM

$$-\frac{d^2u}{dx^2} - u = -x^2 \ , \qquad 0 < x < 1$$

$$u(0) = 0 \ , \qquad u(1) = 0$$

- This was already solved in Chapter 1.
- Exact solution is

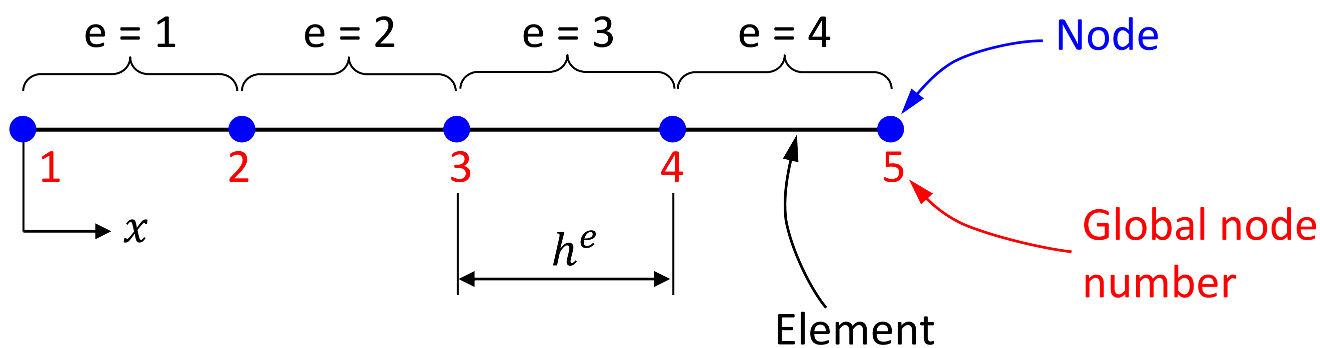
$$u_{exact} = \frac{\sin(x) \ + \ 2\sin(1-x)}{\sin(1)} + x^2 - 2$$



Our First FE Solution (Example 2.1) (cont'd)

- This solution will be very similar to previous Ritz solutions.
- In Chapter 3 we'll make it more algorithmic and easy to program. This will allow us to write our first FE code.

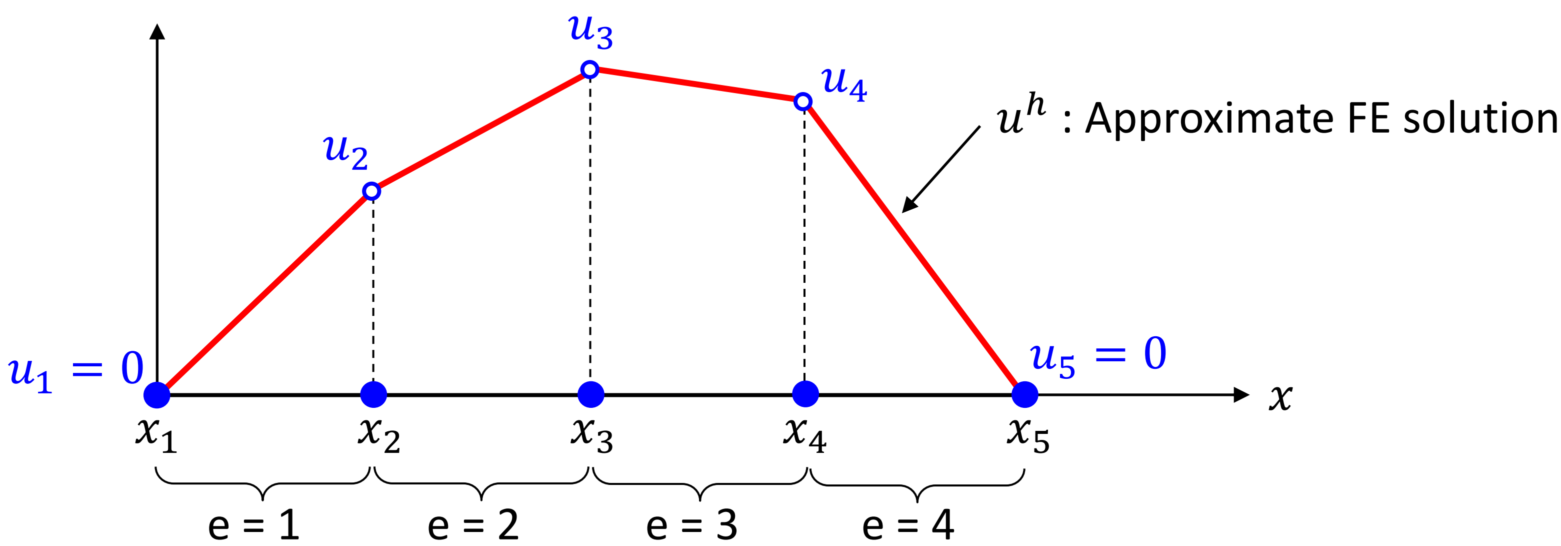
- Following 5 node ($NN = 5$) and 4 element ($NE = 4$) mesh (grid) will be used.



- This is a mesh of linear elements (elements that are defined by 2 nodes).
- This mesh is uniform, i.e. element length $h^e = 0.25$ is constant.

Our First FE Solution (Example 2.1) (cont'd)

- With linear elements we'll obtain a piecewise linear solution



- FE solution is linear over each linear element.
- This solution is C^0 continuous, i.e. it is continuous at element interfaces, but its 1st derivative is not.
- u_j's are the nodal unknown values. The ultimate task is to calculate them.
- u_1 and u_5 are specified as EBCs. They are actually known.

Our First FE Solution (Example 2.1) (cont'd)

- FE solution of the previous slide can be written as

$$u^h = \sum_{j=1}^{NN} u_j \phi_j$$

NN : Number of nodes

u_j : Nodal unknowns

$\phi_j(x)$: Approximation functions

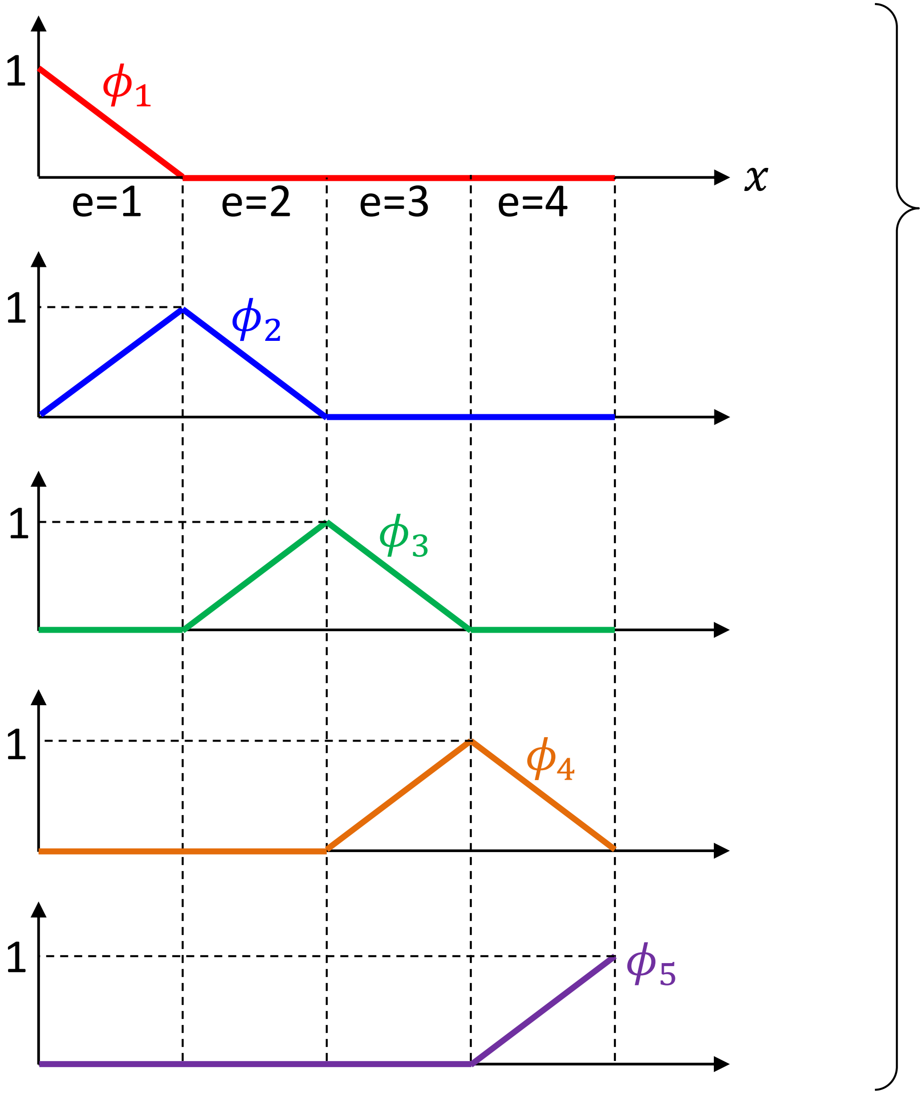
- Same form as Ritz.
- But now unknowns are not just arbitrary numbers. They have a physical meaning, they are the unknown values (e.g. temperatures) at the mesh nodes.
- In FEM ϕ_0 is not necessary.

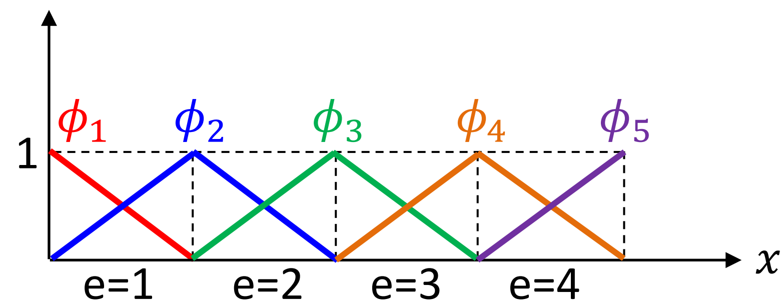
- To have a piecewise linear u^h each ϕ_j should be linear.
- The above sum should provide nodal unknown values at the nodes. This is satisfied if the following Kronecker-Delta property holds

$$\phi_j(x_i) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \quad , \qquad i,j = 1,2,\dots,NN$$

Our First FE Solution (Example 2.1) (cont'd)

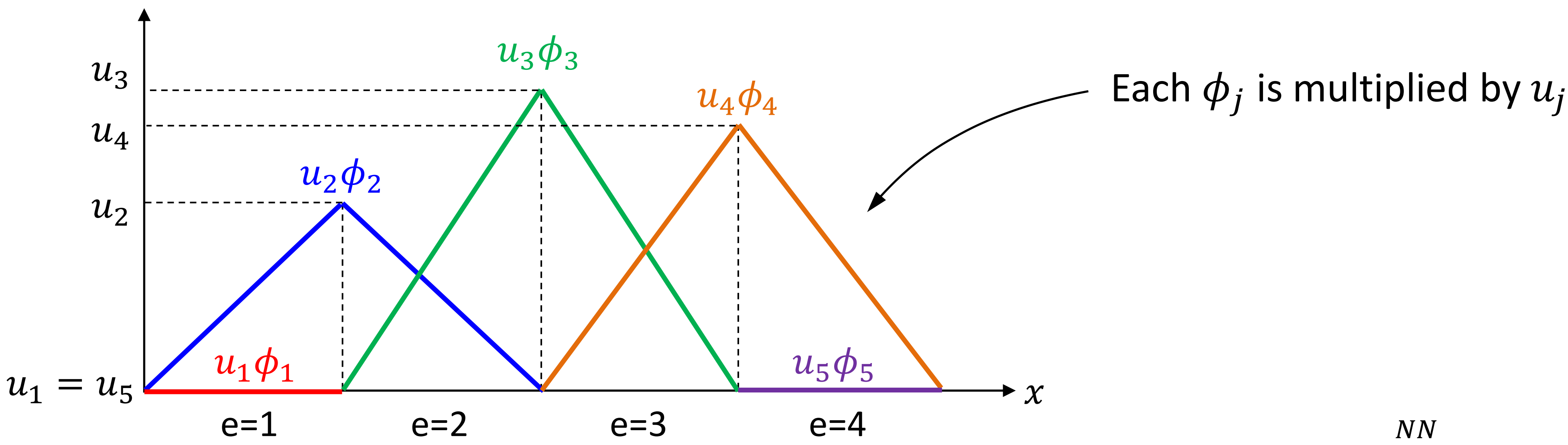
- Following approximation functions will work



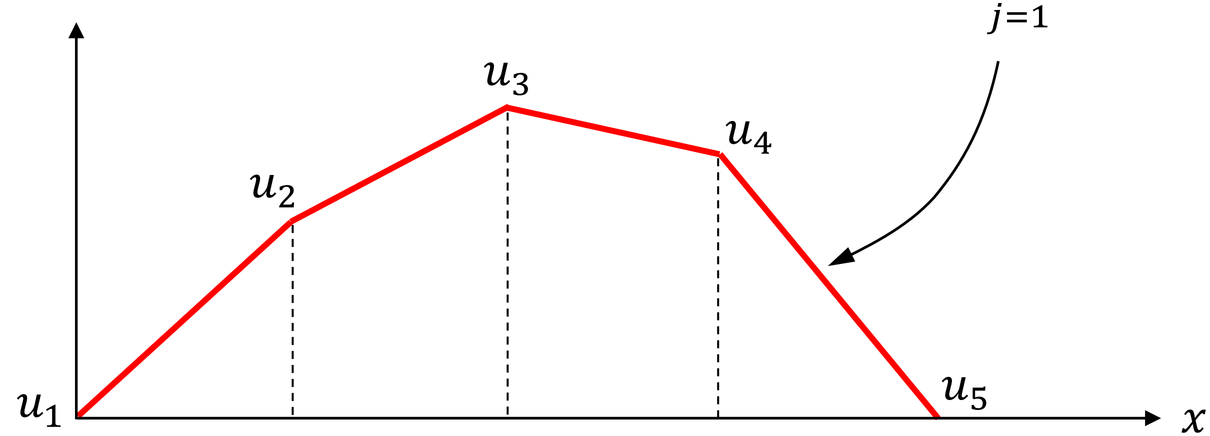


- These are Lagrange type approximation functions.
- They make sure that the solution is continuous across elements, but not its first derivative.
- They have Kronecker-Delta property.
- They have local support, i.e. nonzero only over at most two elements.

Our First FE Solution (Example 2.1) (cont'd)



$$u^h = \sum_{j=1}^{NN} u_j \phi_j$$



Our First FE Solution (Example 2.1) (cont'd)

- FEM uses weak form of the DE (same as Ritz) (u is used instead of u^h for clarity)

$$R = -\frac{d^2u}{dx^2} - u + x^2$$

$$\int_0^1 wR\,dx = 0 \quad \rightarrow \quad \int_{x=0}^1 \left(-w\frac{d^2u}{dx^2} - wu + wx^2\right) dx = 0$$

IBP

$$\int_0^1 \frac{dw}{dx}\frac{du}{dx}dx - \left[w\frac{du}{dx}\right]_0^1$$

Weak form :

$$\int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} - wu + wx^2\right) dx - \left[w\frac{du}{dx}\right]_0^1 = 0$$

Our First FE Solution (Example 2.1) (cont'd)

- We need to write the weak form NN times with NN different w's.
- In Galerkin FEM (GFEM) weight function selection is the same as Ritz

$$w_i = \phi_i \quad , \qquad i = 1,2,\dots,NN$$

1st eqn $(w = \phi_1)$: $\int_0^1 \left(\frac{d\phi_1}{dx}\frac{du}{dx} - \phi_1 u + \phi_1 x^2\right)dx - \left(\underbrace{\phi_1}_{0}\frac{du}{dx}\right)\Big|_{x=1} + \left(\underbrace{\phi_1}_{1}\frac{du}{dx}\right)\Big|_{x=0} = 0$

2nd eqn $(w = \phi_2)$: $\int_0^1 \left(\frac{d\phi_2}{dx}\frac{du}{dx} - \phi_2 u + \phi_2 x^2\right)dx - \left(\underbrace{\phi_2}_{0}\frac{du}{dx}\right)\Big|_{x=1} + \left(\underbrace{\phi_2}_{0}\frac{du}{dx}\right)\Big|_{x=0} = 0$

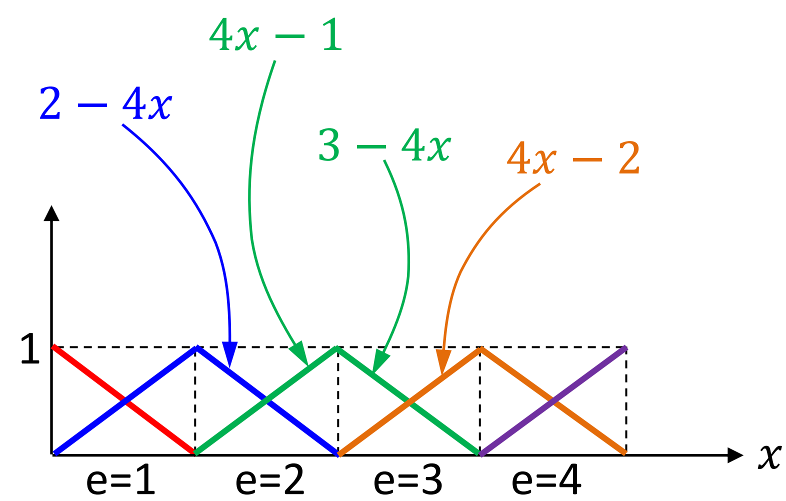
3rd eqn $(w = \phi_3)$: $\int_0^1 \left(\frac{d\phi_3}{dx}\frac{du}{dx} - \phi_3 u + \phi_3 x^2\right)dx - \left(\underbrace{\phi_3}_{0}\frac{du}{dx}\right)\Big|_{x=1} + \left(\underbrace{\phi_3}_{0}\frac{du}{dx}\right)\Big|_{x=0} = 0$

4th eqn $(w = \phi_4)$: $\int_0^1 \left(\frac{d\phi_4}{dx}\frac{du}{dx} - \phi_4 u + \phi_4 x^2\right)dx - \left(\underbrace{\phi_4}_{0}\frac{du}{dx}\right)\Big|_{x=1} + \left(\underbrace{\phi_4}_{0}\frac{du}{dx}\right)\Big|_{x=0} = 0$

5th eqn $(w = \phi_5)$: $\int_0^1 \left(\frac{d\phi_5}{dx}\frac{du}{dx} - \phi_5 u + \phi_5 x^2\right)dx - \left(\underbrace{\phi_5}_{1}\frac{du}{dx}\right)\Big|_{x=1} + \left(\underbrace{\phi_5}_{0}\frac{du}{dx}\right)\Big|_{x=0} = 0$

Our First FE Solution (Example 2.1) (cont'd)

- Integrals
  - are easier to evaluate over each element separately.
  - are nonzero only over certain elements
  - For example 3rd eqn's integral is



$$I_3 = \int_0^1 \left(\frac{d\phi_3}{dx}\frac{du}{dx} - \phi_3 u + \phi_3 x^2 \right) dx$$

which is nonzero only over e=2 and e=3

because ϕ_3 is nonzero only over e=2 and e=3.

$$I_3 = \int_{\Omega^2} (\dots\dots\dots\dots) dx + \int_{\Omega^3} (\dots\dots\dots\dots) dx$$

$$w = \phi_3 = 4x - 1$$

$$u = \sum u_j \phi_j = \underbrace{u_2(2-4x) + u_3(4x-1)}_{\text{Simplified sum over e=2}}$$

$$w = \phi_3 = 3 - 4x$$

$$u = \sum u_j \phi_j = \underbrace{u_3(3-4x) + u_4(4x-2)}_{\text{Simplified sum over e=3}}$$

Our First FE Solution (Example 2.1) (cont'd)

- I_3 can be evaluated in MATLAB as

Part of Example2_1.m code

```
syms x u1 u2 u3 u4 u5;
% First calculate the part of the integral over the 2nd element.
w = 4*x-1;  % w = Phi3 and it is equal to 4x-1 over e=2.
dwdx = diff(w, x);
u = u2*(2-4*x) + u3*(4*x-1);  % This is what u is over e=2
dudx = diff(u, x);
part1 = int(dwdx*dudx - w*u + w*x^2, x, 0.25, 0.5);

% Now calculate the part over the 3rd element.
w = 3-4*x;  % w = Phi3 and it is equal to 3-4x over e=3.
dwdx = diff(w, x);
u = u3*(3-4*x) + u4*(4*x-2);  % This is what u is over e=3
dudx = diff(u, x);
part2 = int(dwdx*dudx - w*u + w*x^2, x, 0.5, 0.75);

% Add two parts to get the integral of the 3rd equation.
I3 = part1 + part2
```

Our First FE Solution (Example 2.1) (cont'd)

- The result for I_3 is : $-\frac{97}{24}u_2 + \frac{47}{6}u_3 - \frac{97}{24}u_4 + \frac{25}{384}$
- Other integrals are calculated in Example2.1v2.m MATLAB code (see the next slide).
- The resultant 5 equations are

$$\underbrace{\begin{bmatrix} \frac{47}{12} & -\frac{97}{24} & & & \\ -\frac{97}{24} & \frac{47}{6} & -\frac{97}{24} & & \\ & -\frac{97}{24} & \frac{47}{6} & -\frac{97}{24} & \\ & & -\frac{97}{24} & \frac{47}{6} & -\frac{97}{24} \\ & & & -\frac{97}{24} & \frac{47}{12} \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}}_{\{u\}} = \underbrace{\begin{Bmatrix} -\frac{1}{768} \\ -\frac{7}{384} \\ -\frac{25}{384} \\ -\frac{55}{384} \\ -\frac{27}{256} \end{Bmatrix}}_{\{F\}} + \underbrace{\begin{Bmatrix} \overbrace{-\left(\frac{du}{dx}\right)\Big|_{x=0}}^{Q_1} \\ 0 \\ 0 \\ 0 \\ \underbrace{\left(\frac{du}{dx}\right)\Big|_{x=1}}_{Q_5} \end{Bmatrix}}_{\{Q\}}$$

Our First FE Solution (Example 2.1) (cont'd)

Example2_1v2.m code (2^{nd} & simpler version)

```
syms x u1 u2 u3 u4 u5 Phi I;
% Phi(i,j) is the i-th approx. funct. over element j
Phi(1,1) = 1-4*x;
Phi(2,1) = 4*x;      Phi(2,2) = 2-4*x;
Phi(3,2) = 4*x-1;    Phi(3,3) = 3-4*x;
Phi(4,3) = 4*x-2;    Phi(4,4) = 4-4*x;
Phi(5,4) = 4*x-3;

coord = [0, 0.25, 0.5, 0.75, 1];

for i=1:5         % Integral loop
   I(i) = 0;      % Initialize the i-th integral to zero.
   for e=1:4      % Element loop
      w = Phi(i,e);
      dwdx = diff(w, x);

      u = u1*Phi(1,e) + u2*Phi(2,e) + u3*Phi(3,e) + u4*Phi(4,e) + u5*Phi(5,e);
      dudx = diff(u, x);

      I(i) = I(i) + int(dwdx*dudx - w*u + w*x^2, x, coord(e), coord(e+1));
   end
end
```

Our First FE Solution (Example 2.1) (cont'd)

$$[K]\{u\} = \{F\} + \{Q\}$$

- Stiffness matrix: $NN \times NN$
- Nodal unknown vector: $NN \times 1$
- Force vector: $NN \times 1$
- Boundary term vector: $NN \times 1$

- This system has 5 equations for 5 unknowns.
- u_1 and u_5 are known, but Q_1 and Q_5 are unknown.
- As a rule, if the PV is known at a boundary, corresponding SV is unknown, and vice versa.
- Note that Q_1 includes a minus sign, which can be thought as an indicator for boundary normal direction.
  - At $x = 0$, boundary outward normal is in $-x$ direction $\rightarrow$ $Q_1 = \left(-\frac{du}{dx}\right)_{x=0}$
  - At $x = 1$, boundary outward normal is in $+x$ direction $\rightarrow$ $Q_5 = \left(+\frac{du}{dx}\right)_{x=1}$

Our First FE Solution (Example 2.1) (cont'd)

- In practice we first want to solve for the unknown u's, but not Q's.
- For this we apply reduction to the $NN \times NN$ system and drop the 1st and 5th equations, because u_1 and u_5 are known.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{25} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix}$$

- The reduced system is 3x3

$$\begin{bmatrix} K_{22} & K_{23} & K_{24} \\ K_{32} & K_{33} & K_{34} \\ K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_2 - K_{21}u_1 - K_{25}u_5 \\ F_3 - K_{31}u_1 - K_{35}u_5 \\ F_4 - K_{41}u_1 - K_{45}u_5 \end{Bmatrix} + \begin{Bmatrix} Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

Our First FE Solution (Example 2.1) (cont'd)

- Reduced system for this problem is

$$\begin{bmatrix} \frac{47}{6} & -\frac{97}{24} & 0 \\ -\frac{97}{24} & \frac{47}{6} & -\frac{97}{24} \\ 0 & -\frac{97}{24} & \frac{47}{6} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -\frac{7}{384} - 0 - 0 \\ -\frac{25}{384} - 0 - 0 \\ -\frac{55}{384} - 0 - 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

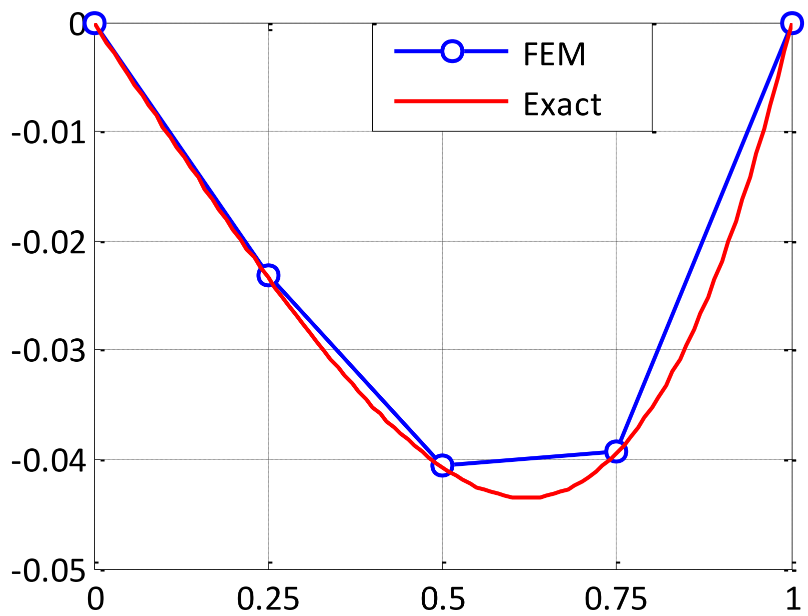
Because $u_1 = 0$, $u_5 = 0$

- Solving this system gives $\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -0.0232 \\ -0.0405 \\ -0.0392 \end{Bmatrix}$

- Exact values are $\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix}_{exact} = \begin{Bmatrix} -0.0234 \\ -0.0408 \\ -0.0394 \end{Bmatrix}$

Our First FE Solution (Example 2.1) (cont'd)

```
coord = [0 0.25 0.5 0.75 1];
u = [0 -0.0232 -0.0405 -0.0392 0];
plot(coord, u, 'o-b', 'LineWidth', 2)
hold
x = 0:0.01:1;
uExact = (sin(x) + 2*sin(1-x)) / sin(1) + x.*x - 2;
plot(x, uExact, 'r', 'LineWidth', 2)
grid on
```



4 element FE solution
vs.
exact solution

Our First FE Solution (Example 2.1) (cont'd)

- Q_1 and Q_5 values generally have physical meaning, such as heat flux or reaction force.
- After calculating PVs, these SVs can be calculated in two ways.

- First way : Use the 1st and 5th eqns of slide 2-14

$$Q_1 = \frac{47}{12}u_1 - \frac{97}{24}u_2 + \frac{1}{768} = 0.0951$$

$$Q_5 = -\frac{97}{24}u_4 + \frac{47}{12}u_5 + \frac{27}{256} = 0.2639$$

- Second way : Use the derivatives of the FE solution at the boundaries

$$Q_1 = -\left(\frac{du_h}{dx}\right)\Bigg|_{x=0} = -\frac{u_2 - u_1}{h^e} = -\frac{-0.0232 - 0}{0.25} = 0.0928$$

$$Q_5 = \left(\frac{du_h}{dx}\right)\Bigg|_{x=1} = \frac{u_5 - u_4}{h^e} = \frac{0 - (-0.0392)}{0.25} = 0.1568$$

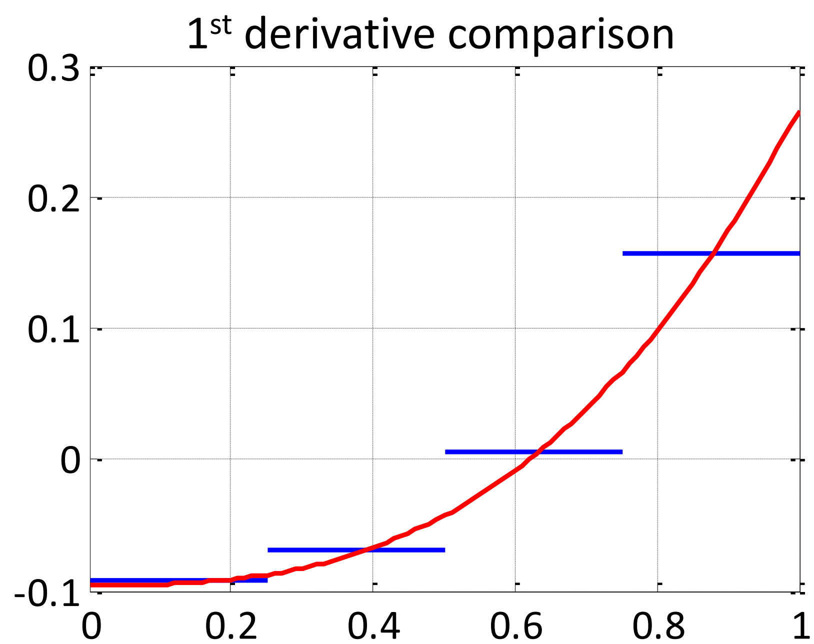
Good match (between the two Q_1 values)

No match (between the two Q_5 values)
Why?
Which one is better?

Our First FE Solution (Example 2.1) (cont'd)

- FEM provides very good nodal values, but what about the first derivative?

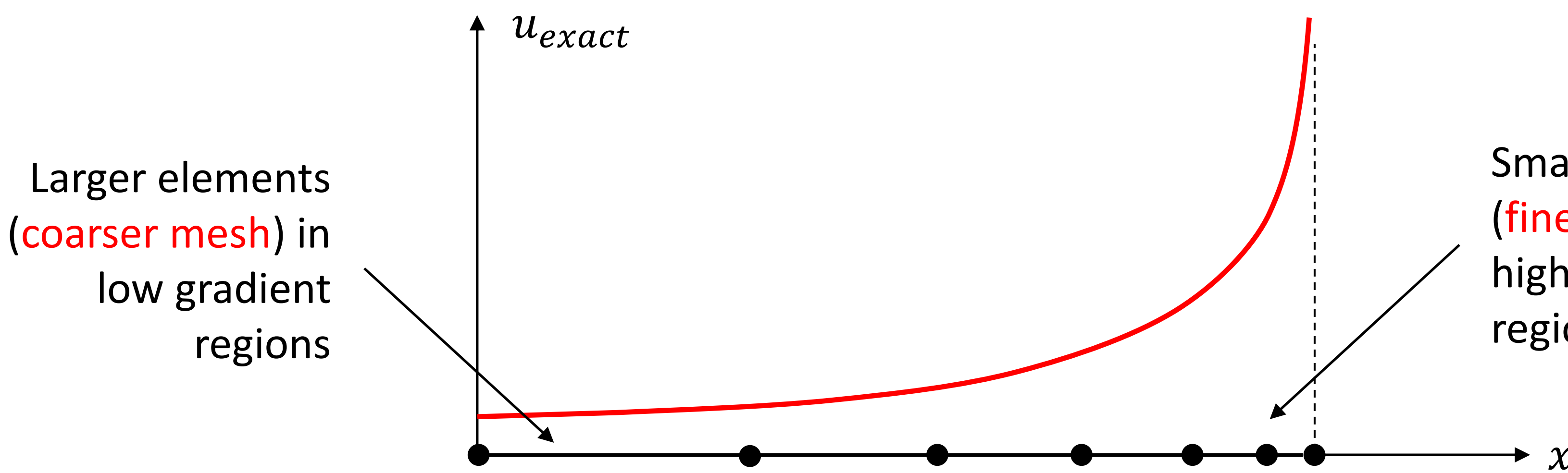
```
coord = [0 0.25 0.5 0.75 1];
u = [0 -0.0232 -0.0405 -0.0392 0];
for i = 1:4
   slope(i) = (u(i+1)-u(i)) / 0.25;
end
figure; hold
for i = 1:4
   plot([coord(i) coord(i+1)], ...
        [slope(i) slope(i)], ...
        '-b', 'LineWidth', 2)
end
syms x;
duExact = diff((sin(x) + 2*sin(1-x)) / sin(1) + x.*x - 2, x);
X = 0:0.01:1;
plot(X, subs(duExact,x,X), 'r', 'LineWidth', 2)
grid on
```



Remarks About Our First FEM Solution

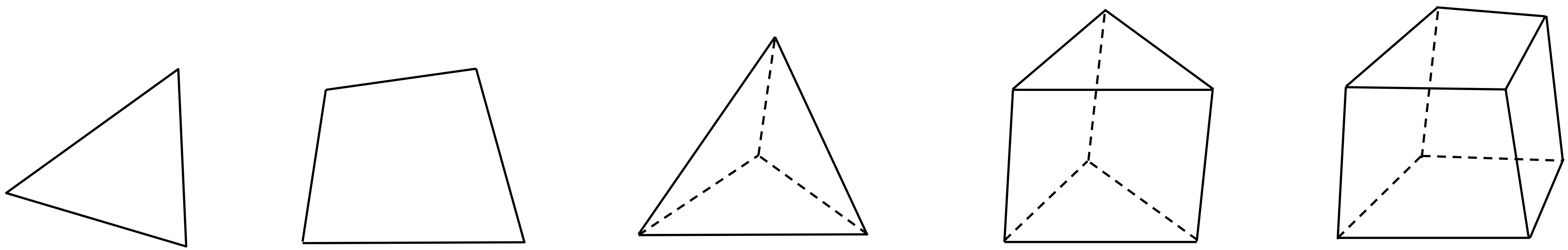
- The procedure is not suitable to computer programming.
  - Approximation function selection must be made mesh independent.
  - Symbolic integration is costly and not readily available for many prog. languages.
  - Numerical integration is used in FEM codes.

- For real problems non-uniform meshes are preferred.
  - Generating a good mesh is not easy. Adaptive Mesh Refinement (AMR) is helpful here.



Remarks (cont’d)

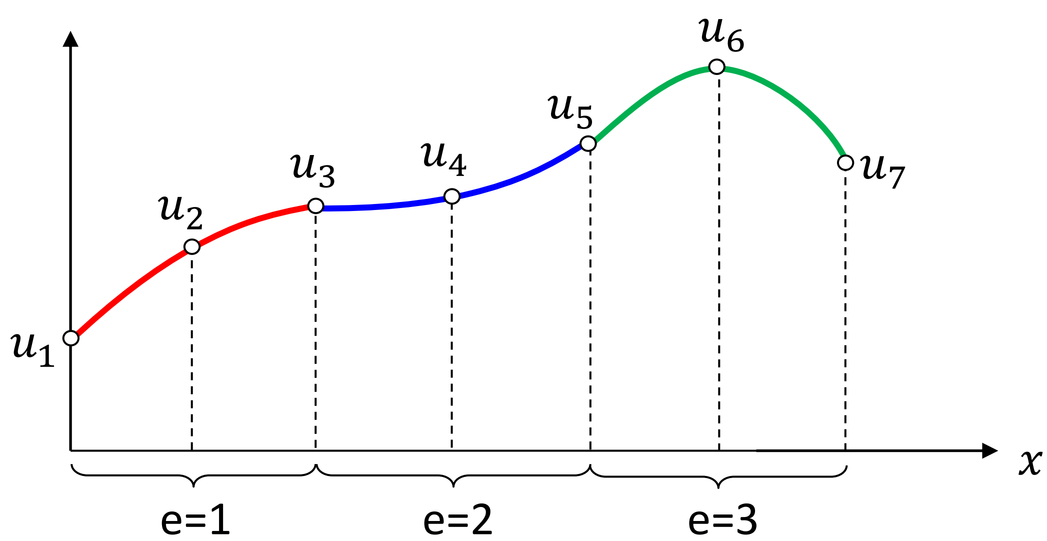
- 2D elements can be triangular or quadrilateral.
- In 3D they can be tetrahedral, triangular prism (wedge) or hexahedral (brick).



Triangular | Quadrilateral | Tetrahedron (Pyramid) | Triangular prism (Wedge) | Hexahedron (Brick)

Remarks (cont'd)

- In our first FE solution we used linear (2-node) elements.
- It is possible to use higher order elements, which make use of more nodes.
- When we use quadratic (3-node) elements we can have quadratic polynomial solutions over each element.
- The following sample solution makes use of 3 quadratic elements ($NE = 3$) with a total node number of 7 ($NN = 7$).



Remarks (cont'd)

- Our first FE solution was piecewise continuous.
- It was continuous across element interfaces, however its first deriative was not.
- This is known as a C^0 continuous solution, i.e. only the 0th derivative of the unknown (which is the unknown itself) is continuous.
- For 2nd order DEs, the weak form contains first derivative of the unknown and the use of C^0 continuous solution is enough.
- For higher order DEs, for example the 4th order one used for beam bending, C^0 continuity is not enough and a C^1 continuous solution is necessary.

- FEM results in a sparse stiffness matrix, i.e. a matrix with lots of zero entries.
- This is due to the compact support property of the approximation functions.
- For 2D and especially for 3D problems this feature is important to decrease memory usage of the code.

Remarks (cont'd)

- The structure of the final linear algebraic equation system depends on how we number the mesh nodes globally.
- When we do a different numbering, the final equation system does NOT change mathematically, however the $[K]$ matrix changes.
- In certain linear system solution techniques $[K]$ is stored as a banded matrix and bandwidth of $[K]$ is tried to be minimized to decrease memory usage.
- Bandwith of $[K]$ is directly related to how we number the mesh nodes. Commercial FE software use bandwidth reduction algorithms to minimize the bandwidth of $[K]$.

- In the problem we solved, $[K]$ turns out to be symmetric.
- This is a due to the solved DE and the use of GFEM.
- The following DE with the additional 1st derivative will not result in a symmetric $[K]$.

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} - u = -x^2$$

Remarks (cont'd)

- The DE we solved was linear.
- For a nonlinear DE, such as the following one, an extra linearization step is necessary to obtain the system of linear algebraic equations.

$$-u\frac{d^2u}{dx^2} - u = -x^2$$